

Undecidable Classical Properties of Observers

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A property of a system is called actual, if the observation of the outcome of the test that pertains to that property yields an affirmation with certainty. We formalize the act of observation by assuming the outcome itself is an actual property of the state of the observer after the act of observation and correlates with the state of the system. For an actual property this correlation needs to be perfect. A property is called classical if either the property or its negation is actual. We show by a diagonal argument that there exist classical properties of an observer that he cannot observe perfectly. Because states are identified with the collection of properties that are actual for that state, it follows no observer can perfectly observe his own state. Implications for the quantum measurement problem are briefly discussed.

KEY WORDS: undecidability; observation; self-reference; quantum measurement problem.

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1. INTRODUCTION: STATES AND PROPERTIES

In the realm of quantum physical experiments, data often consists of nothing more than clicks of detectors or spots on a photographic plate collected at certain instances and positions. With a sufficient amount of such clicks, obtained under specific circumstances, we can establish the properties and hence determine the state of the system, at least to a precision that depends in principle only on the amount of clicks we care to gather. How does this work? The state, a representation of the “mode of being” of a system, determines the properties of the system. Loosely speaking, a property of a system is something that can be attributed to that system with a certain persistency. Birkhoff and von Neumann made the first decisive step toward the characterization of a system through its properties. They called quantities which yield only one of two possible results, and which are testable in a reproducible sense “experimental propositions.” Such experiments also form the operational basis of the Geneva–Brussels approach, in which a

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property is introduced as an equivalence class of tests (also called questions or experimental projects), and a state is regarded as a set of actual properties. We will introduce the notion of property and state through the state-property spaces as can be found in Aerts (1999), but this paper is fully self-contained and we will introduce the required definitions along the way. Consider a physical entity S with its (nonempty) set of states Σ_S and (nonempty) set of properties \mathcal{L}_S . States will be denoted by small roman script p, q, r, \dots and properties by small bold script $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$. We assume the state fully characterizes the properties of the entity, some of which may be actual, and some not. What do we mean when we say an entity has an actual property \mathbf{a} ? We assume that for every property \mathbf{a} in \mathcal{L}_S there exists a test, let us call it $\hat{\chi}_{\mathbf{a}}$, that tests property \mathbf{a} . The result of the test is a simple “yes” or “no.”² If the test yields “yes” with *certainty*, we say property \mathbf{a} is *actual*.

Definition 1 (Actual Property). An entity S in the state p , is said to have an *actual property* \mathbf{a} iff the test $\hat{\chi}_{\mathbf{a}}$ that tests property \mathbf{a} yields “yes” with certainty.

Note that this is in accordance with what we intuitively mean when we say that a system has a property. Two tests $\hat{\chi}_{\mathbf{a}}$ and $\hat{\chi}_{\mathbf{b}}$ are called *equivalent* when it is the case that for any state p where $\hat{\chi}_{\mathbf{a}}$ gives with certainty “yes,” we also have that $\hat{\chi}_{\mathbf{b}}$ gives with certainty “yes,” and vice versa. We will say equivalent tests, test the same property. Vice versa, a property is identified with an equivalence class of tests. A property is actual when any test in the equivalence class yields “yes” with certainty, because all tests will then yield “yes” with certainty.

A given property $\mathbf{a} \in \mathcal{L}_S$ may be actual for some of the states $p \in \Sigma_S$ of the entity, but not necessarily for all. To make this notion precise, employing the usual notation $\mathcal{P}(\Sigma_S)$ for the set of all subsets of Σ_S , we postulate the map $\kappa_S : \mathcal{L}_S \rightarrow \mathcal{P}(\Sigma_S)$, called the Cartan map, such that $\kappa_S(\mathbf{a})$ is the set of states in Σ_S for which the property \mathbf{a} is actual. The triple $(\Sigma_S, \mathcal{L}_S, \kappa_S)$ will be called a *state property space* (Aerts, 1999) and fully characterizes what can be known about the entity with certainty.

Definition 2 (State Property Space). The triple $(\Sigma_S, \mathcal{L}_S, \kappa_S)$, called a *state property space*, consists of two sets Σ_S and \mathcal{L}_S (where Σ_S is the set of states of a physical entity S , and \mathcal{L}_S its set of properties), and a function $\kappa_S : \mathcal{L}_S \rightarrow \mathcal{P}(\Sigma_S)$, such that $\mathbf{a} \in \mathcal{L}_S$ is actual for the entity in a state p iff $p \in \kappa_S(\mathbf{a})$.

When a property is not actual for the entity S in a given state, we will say this property is *potential*.

Definition 3 (Potential Property). If, for a given entity S in the state p , the property $\mathbf{b} \in \mathcal{L}_S$ is not actual ($p \notin \kappa_S(\mathbf{b})$), then the property \mathbf{b} is called *potential* for the entity S in the state p . We write $p \in \kappa_S^C(\mathbf{b}) \equiv \Sigma_S - \kappa_S(\mathbf{b})$.

²If one prefers a mathematical definition, one can regard the test as a mapping $\Sigma_S \times \mathcal{L}_S \rightarrow \{\text{yes}, \text{no}\}$. This is not customary in the Geneva–Brussels approach, where the test is mainly regarded as an operational primitive to come to the formal description provided by the Cartan map and its inverse.

Operationally speaking, there is an obvious inverse of a test that we introduce in the following definition.

Definition 4 (Inverse Test). If $\hat{\chi}_a$ tests property \mathbf{a} , then switching the roles of “yes” and “no” defines a new test denoted $\hat{\chi}_{a^\perp}$, which yields “yes” when $\hat{\chi}_a$ yields “no,” and vice versa. We will call $\hat{\chi}_{a^\perp}$ the *inverse test* of $\hat{\chi}_a$.

The test $\hat{\chi}_{a^\perp}$ is operationally well defined by simply switching the outcomes of the test $\hat{\chi}_a$. The notation $\hat{\chi}_{a^\perp}$ suggests that it tests “the property \mathbf{a}^\perp ,” but in general we cannot associate \mathbf{a}^\perp with a well-defined property as an equivalence class of tests. The problem is that one can easily give examples of tests that are equivalent to $\hat{\chi}_a$, but for which the inverse test is not equivalent to $\hat{\chi}_{a^\perp}$.³ There is however an important class of properties, the classical properties, for which this problem does not arise.

Definition 5 (Classical Property). A test that has a predetermined answer, is a classical test. A property that is defined as an equivalence class of classical tests, will be called a *classical property*.

Definition 6 (Inverse Classical Property). Given an entity S with a classical property $\mathbf{a} \in \mathcal{L}_S$. Then the inverse test $\hat{\chi}_{a^\perp}$, defines a property denoted \mathbf{a}^\perp , called the *inverse* of \mathbf{a} .

To show that this definition makes sense, we show that for a classical property, the inverse property \mathbf{a}^\perp is well defined by the equivalence class of tests that contains $\hat{\chi}_{a^\perp}$. That is, we have to show that for two arbitrary tests $\hat{\chi}_a$ and $\hat{\chi}'_a$ in the equivalence class of a classical property \mathbf{a} , the corresponding inverse tests, $\hat{\chi}_{a^\perp}$ and $\hat{\chi}'_{a^\perp}$, are also equivalent. First note that, because $\hat{\chi}_a$ and $\hat{\chi}'_a$ are classical tests, so are their inverses, hence all tests give either “yes” or “no” with certainty. Suppose that, for a given entity in a given state, $\hat{\chi}_{a^\perp}$ gives “yes,” then $\hat{\chi}_a$ gives “no.” Since $\hat{\chi}_a$ and $\hat{\chi}'_a$ are equivalent by assumption, $\hat{\chi}'_a$ would give “no” too. But then $\hat{\chi}'_{a^\perp}$ gives “yes,” so that $\hat{\chi}_{a^\perp}$ and $\hat{\chi}'_{a^\perp}$ are equivalent, and \mathbf{a}^\perp is well defined.

Obviously, if the property \mathbf{a} is classical, then so is \mathbf{a}^\perp . Equally obvious, we have for a classical property that the operation of inversion is idempotent: $\mathbf{a}^{\perp\perp} = \mathbf{a}$. In the light of the preceding remarks, it is natural to postulate that, for an entity S for which $\mathbf{a} \in \mathcal{L}_S$ is a classical property, we always have that $\mathbf{a}^\perp \in \mathcal{L}_S$ too. For an entity S that has a classical property \mathbf{a} , we have that $\forall s \in \Sigma_S$ either \mathbf{a} , or \mathbf{a}^\perp is

³ As an example, take a spin 1 system, for which the spin S can take either one of three values $\{-1, 0, +1\}$. Call \mathbf{a} the property of yielding $S = +1$ with certainty when tested, and likewise \mathbf{b} and \mathbf{c} the properties of having $S = 0$ and $S = -1$, respectively. Because the actuality of the properties \mathbf{a} and \mathbf{b} mutually exclude each other, we could perhaps propose $\mathbf{b} = \mathbf{a}^\perp$. By the same token, \mathbf{c} is a candidate for \mathbf{b}^\perp . However, the tests pertaining to \mathbf{a} and \mathbf{c} are manifestly not equivalent.

actual. Hence a classical property \mathbf{a} partitions the state space Σ_S in just two sets

$$\begin{aligned}\kappa_S(\mathbf{a}) \cup \kappa_S(\mathbf{a}^\perp) &= \Sigma_S \\ \kappa_S(\mathbf{a}) \cap \kappa_S(\mathbf{a}^\perp) &= \emptyset\end{aligned}\tag{1}$$

We note that in the development of the Geneva–Brussels approach, further axioms are imposed on \mathcal{L}_S so that the properties of a general physical system form a complete, atomistic and orthomodular lattice (Aerts, 1999). The problem of the “inverse property” is then treated by the introduction of ortho-axioms on \mathcal{L}_S (Aerts and Deses, 2005), and the role of the inversion is played by the orthocomplementation. For a Boolean sublattice of \mathcal{L}_S , or for an entity that is classical, (i.e., an entity for which \mathcal{L}_S is a Boolean lattice), the orthocomplementation reduces to the Boolean NOT. In either case, it is assumed that, if $\mathbf{a} \in \mathcal{L}_S$, then $\mathbf{a}^\perp \in \mathcal{L}_S$ too. For our purposes here, we will only require the existence of the inverse of every *classical* property in \mathcal{L}_S , as established operationally through the definition of the inverse test.

2. THE FORMALIZATION OF OBSERVATION

Up to now, we are in close accordance with the Geneva–Brussels approach (Aerts, 1999), which is recognized for being both realistic (entities *are* in a state that completely describes the status of all properties that pertain to the entity) and operational (properties are defined as equivalence classes of experiments). However, the question of *how* to test a property, is not formalized. According to the Geneva–Brussels prescription, to see whether property \mathbf{a} is actual, an observer needs to perform the test $\hat{\chi}_\mathbf{a}$. But if states are indeed realistic descriptions of systems, then in a more detailed account, the observer has to be regarded as a system having properties in its own right. We can define the state property space for the observer just as we did for the entity: $(\Sigma_M, \mathcal{L}_M, \kappa_M)$. The observing system performs the test and formulates the outcome. This outcome defines in a natural way an actual property of the state of the observer after the observation. Indeed, *if* an outcome occurs, the state of the observer has either the property that M formulates the outcome “yes,” or M formulates the outcome “no.” We define the property \mathbf{i} as follows:

\mathbf{i} : the outcome indicated by the observing system is “yes”

If the experiment is not repeated and we re-read the outcome of the observation, then we need to get the same result. So the outcome of a test defines a *classical* property of the observer. The particular outcome of an observation then depends on whether the state of the observer after the measurement interaction belongs to

$$\kappa_M(\mathbf{i}) \text{ or to } \kappa_M(\mathbf{i}^\perp)\tag{2}$$

If we call Σ'_M the space of observer states after interaction, then the indicator property partitions $\Sigma'_M : \kappa_M(\mathbf{i}) \cap \kappa_M(\mathbf{i}^\perp) = \emptyset$ and $\kappa_M(\mathbf{i}) \cup \kappa_M(\mathbf{i}^\perp) = \Sigma'_M$. When the observing system is for example a photomultiplier or a Geiger–Muller counter, we even have $\Sigma'_M = \Sigma_M$ as long as the experiment is running. In an arbitrary small time interval of the running experiment, the detector has fired, or it hasn't. Hence its state indicates “yes” or “no” for that interval. This shows that the set of states the active detector can attain, equals the set of states that indicate a “yes” or a “no.”

2.1. The Observer Interacts to Test a Property

So far we have described that a system can have properties, which partitions its set of states, and the observer is the one who indicates the outcome, which in turn partitions his set of states (2). These are two very basic desiderata of the process of observing a property. But we have yet to include the most important ingredient: the result of the observation should pertain to the entity under study. If the observer is faithfully observing the result of the test corresponding to an actual property \mathbf{a} of the entity S , then *a fortiori* the observation has to yield “yes” when the property holds. Hence the state of the observer m' after the act of observation has to express “yes”:

$$s \in \kappa_S(\mathbf{a}) \Rightarrow m' \in \kappa_M(\mathbf{i}) \tag{3}$$

If the observer did not interact with the entity, this implication cannot reasonably be expected to hold. Hereafter, we denote the state of the system S under investigation by s and the state of the observer M that is measuring S , by m . Assume then an observer in a state $m \in \Sigma_M$ interacts with an entity in the state $s \in \Sigma_S$. There are many ways conceivable to form a new state from two interacting states, but we need not go into details. We only assume that the compound state is a function τ of the two constituent states:

$$\tau : \Sigma_S \times \Sigma_M \rightarrow \Sigma_{S+M} \tag{4}$$

From this compound system, there has to be a flow of information to the state of the observer. After the interaction the state of the observer (regarded again as a single system) should convey the outcome of the observation. That is, there has to exist a restriction ρ of the state of the total system to the set of states of the observer

$$\rho : \Sigma_{S+M} \rightarrow \Sigma_M \tag{5}$$

The nature of the restriction ρ is also quite irrelevant for our purposes. It can be a partial trace, or perhaps a projection onto some subspace. It is this new state m' of the observer that indicates “yes” or “no,” depending on whether it belongs to

$\kappa_M(\mathbf{i})$ or to $\kappa_M(\mathbf{i}^\perp)$. We formulate this as $\phi : \Sigma_M \rightarrow \{\text{yes, no}\}$

$$\begin{aligned}\phi(m') = \text{yes} &\iff m' \in \kappa_M(\mathbf{i}) \\ \phi(m') = \text{no} &\iff m' \in \kappa_M(\mathbf{i}^\perp)\end{aligned}\quad (6)$$

We summarize our model of observation by composing (6), (5) and (4) into a single mapping $o = \phi \circ \rho \circ \tau$:

$$o : \Sigma_S \times \Sigma_M \rightarrow \{\text{yes, no}\} \quad (7)$$

We assume that $o(s, m)$ is surjective (both “yes” and “no” can be obtained by an interaction between S and M), and that $o(s, m)$ is defined for all⁴ of $\Sigma_S \times \Sigma_M$. Besides these two relatively mild regularity conditions, it is crucial that o is single-valued for the couples (s, m) such that m is a state that is to measure a property that is *actual* for the system in the state s . Otherwise the implication (3) cannot hold, because actuality means the corresponding test yields “yes” with certainty. For a potential property the outcome can be either “yes” or “no,” so that in this case $o(s, m)$ could be two-valued in principle. Hence we propose the following definition:

Definition 7 (Perfect Observation). Let $\Lambda \in \mathcal{P}(\mathcal{L}_S)$ be a nonempty collection of properties of an entity S . A state m of an observer M , observing an entity S , will be called Λ -perfect iff there exists a mapping (7) $o : \Sigma_S \times \Sigma_M \rightarrow \{\text{yes, no}\}$, such that for each property $\mathbf{a} \in \Lambda$ and for every state $s \in \Sigma_S$, we have that for that particular state $m \in \Sigma_M$:

$$s \in \kappa_S(\mathbf{a}) \implies o(s, m) = \text{yes} \quad (8)$$

$$s \in \kappa_S^C(\mathbf{a}) \implies o(s, m) \text{ could be yes and could be no} \quad (9)$$

We remark that the particular state m of a Λ -perfect observer M for which the correlation (3) holds, will also be called Λ -perfect. This means that an observer M can be Λ -perfect only if there exists at least a single state $m \in \Sigma_M$ that is Λ -perfect, and that he is Λ -perfect only if he is in the state that realizes (3). Moreover, if the state m is Λ -perfect with $\mathbf{a} \in \Lambda$ and we want to concentrate on this specific property \mathbf{a} , then, with slight abuse of notation, we will write “ m is \mathbf{a} -perfect.” We argue that this is the most simple case that deserves to be called perfect observation: when the observer is in a state that is able to properly observe *at least a single property* of an entity, *regardless* of the particular state the entity is in. Suppose we made the notion of Λ -perfectness dependent on the state of S , then the fixed observer, indicating a permanent “yes” (or a “no”), would be “perfect” in perhaps as much as half the cases, without even having to bother about making

⁴We feel justified in this assumption, not because all interactions necessarily lead to well-defined outcomes, but because outcomes that are not defined, cannot express a scientific value.

an observation. We argue that the purpose of observation is to infer the state from the outcomes. The observer cannot, in general, be expected to know in advance which state he is measuring, and “perfectness” entails that he produces the right correlation (3) for *any* state of the entity in its proper state space Σ_S . Of course, different states m and m' can be perfect with respect to different properties and in this way an observer with a rich state space can be perfect with respect to a large set of properties, provided he knows how to attain those states.

Our definition of perfect observation leaves some unnecessary ambiguity with respect to classical properties. Recall that a property is *classical*, if for any state s of S , we have that either \mathbf{a} or \mathbf{a}^\perp is actual. Suppose that \mathbf{a}^\perp is actual, then $s \in \kappa_S^C(\mathbf{a})$ and the definition of perfectness we have now, only tells us the outcome *could* be yes or *could* be no. But the Geneva–Brussels approach tells us more. First, the result of the test that corresponds to a classical property is predetermined, so for a given state s , we have that $o(s, m)$ is always no *or* is always yes. Second, the inverse test $\hat{\chi}_{\mathbf{a}^\perp}$ was defined by switching the outcomes of the test $\hat{\chi}_{\mathbf{a}}$, so we know the result has to be “no” with certainty. Because we formalized the notion of the observation by means of interaction with an observer, we need to formalize this “switching” procedure to test the inverse of a classical property on the level of the state of the observer.

Definition 8 (Inverse Complete). A mapping $\alpha : \Sigma_M \rightarrow \Sigma_M$ is called an *inversion* (with respect to o) iff

$$o(s, m) \neq o(s, \alpha(m))$$

$$o(s, m) = o(s, \alpha(\alpha(m)))$$

A Λ -perfect observer M is called *inverse complete* iff, for every classical property $\mathbf{a} \in \Lambda$ of S in the state $s \in \Sigma_S$, and for every $m \in \Sigma_M$ that tests \mathbf{a} perfectly, there exists at least one state $\alpha(m) \in \Sigma_M$.

Logically speaking, and this is one of the cornerstones in our argument, it makes no sense to assert that we have observed a classical property to be actual, when we know that we would have received the same outcome when the property is not actual. So we argue that the actuality of inverse properties, requires inverse outcomes: for an observer to be perfect with respect to a classical property \mathbf{a} , there has to exist a state of the observer that is \mathbf{a}^\perp -perfect.

Definition 9 (Classical Perfect). A state $m \in \Sigma_M$ of M is \mathbf{a} -classical perfect iff m is \mathbf{a} -perfect and $\alpha(m)$ exists in Σ_M which is \mathbf{a}^\perp -perfect. An observer M will be called *\mathbf{a} -classical perfect* iff he is in a state $m \in \Sigma_M$ that is \mathbf{a} -classical perfect.

But if an observer is \mathbf{a} -perfect and he is able to switch the roles of “yes” and “no,” then he should also be \mathbf{a}^\perp -perfect. We now show this is indeed the case.

Theorem 1. *Given an entity S with a classical property $\mathbf{a} \in \mathcal{L}_S$, and an inverse complete observer M with an inversion α . Then*

$$m \text{ is } \mathbf{a}\text{-perfect} \iff \alpha(m) \text{ is } \mathbf{a}^\perp\text{-perfect}$$

Proof: We first prove the left-to-right implication. Because $\mathbf{a} \in \mathcal{L}_S$ is a classical property, we have that $\{\kappa_S(\mathbf{a}), \kappa_S(\mathbf{a}^\perp)\}$ is a partition of Σ_S . Suppose then first that $s \in \kappa_S(\mathbf{a})$. If m is \mathbf{a} -perfect, then $o(s, m) = \text{yes}$. Hence $o(s, \alpha(m)) = \text{no}$, indicating $\alpha(m)$ is indeed \mathbf{a}^\perp -perfect when $s \in \kappa_S(\mathbf{a})$. Suppose on the other hand that $s \in \kappa_S(\mathbf{a}^\perp)$. By the assumption that m is \mathbf{a} -perfect, with \mathbf{a} classical, we have that $o(s, m)$ is single-valued and $o(s, m)$ is always yes or is always no. By the definition of the inversion α , we then get $o(s, \alpha(m))$ is always no or is always yes. Suppose then that $o(s, \alpha(m))$ is always no. Then $o(s, \alpha(\alpha(m)))$ is always yes, and by the idempotency of α , $o(s, m)$ is also always yes. But if $o(s, m)$ is always yes, then (because m is \mathbf{a} -perfect by assumption) this implies $s \in \kappa_S(\mathbf{a})$, which contradicts the assumption that $s \in \kappa_S(\mathbf{a}^\perp)$. Therefore $o(s, \alpha(m))$ cannot be always no, and hence has to be always yes. We then have $s \in \kappa_S(\mathbf{a}^\perp) \implies o(s, \alpha(m)) = \text{yes}$, making $\alpha(m)$ indeed \mathbf{a}^\perp -perfect when $s \in \kappa_S(\mathbf{a}^\perp)$. For the reverse implication, call $\mathbf{b} = \mathbf{a}^\perp$, and $\alpha(m) = m^*$. Because \mathbf{a} is classical, so is \mathbf{b} . Then, by the first part of the proof, we have m^* is \mathbf{b} -perfect implies $\alpha(m^*)$ is \mathbf{b}^\perp -perfect. By idempotency of α we have $o(s, \alpha(m^*)) = o(s, \alpha(\alpha(m))) = o(s, m)$ and for a classical property $\mathbf{b}^\perp = \mathbf{a}^{\perp\perp} = \mathbf{a}$. So m is \mathbf{a} -perfect. \square

As the inversion α is defined on the level of the outcomes generated by $o(s, m)$, the image $\alpha(m)$ need not necessarily be unique. Still the last theorem can be reversed, as is shown in the following:

Theorem 2. *Given an entity S with a classical property $\mathbf{a} \in \mathcal{L}_S$ and an observer M . If $m \in \Sigma_M$ is \mathbf{a} -perfect and $m^* \in \Sigma_M$ is \mathbf{a}^\perp -perfect, then m^* is an inversion of m .*

Proof: Both m and m^* are in Σ_M , hence $m^* = \beta(m)$ with $\beta : \Sigma_M \rightarrow \Sigma_M$. If m is \mathbf{a} -perfect and $s \in \kappa_S(\mathbf{a})$, then $o(s, m)$ is always yes. Hence if $\beta(m)$ is perfectly observing \mathbf{a}^\perp then $o(s, \beta(m))$ can never be yes because \mathbf{a} is classical. Hence we have $o(s, m) \neq o(s, \beta(m))$. If on the other hand m is \mathbf{a} -perfect but $s \in \kappa_S(\mathbf{a}^\perp)$, then $o(s, m) = \text{no}$. But by assumption $\beta(m)$ is \mathbf{a}^\perp -perfect and this is actual, hence $o(s, \beta(m)) = \text{yes}$. Because $o(s, m)$ is defined for all of $\Sigma_S \times \Sigma_M$, and can only take the values ‘yes’ and ‘no,’ the condition $o(s, m) \neq o(s, \beta(m))$ implies $o(s, m) = o(s, \beta(\beta(m)))$. So β is indeed an inversion. \square

The correlation (3) thus becomes much stronger for a classical property observed by a classical-perfect observer. Indeed, if \mathbf{a} is classical, we have that either \mathbf{a} or \mathbf{a}^\perp is actual. So either $o(s, m)$ or $o(s, \alpha(m))$ has to be single-valued for

any given s . But because $\alpha(m)$ and m are each other's inversion, both $o(s, \alpha(m))$ and $o(s, m)$ have to be single-valued. So \mathbf{a} -classical perfect observation implies

$$s \in \kappa_S(\mathbf{a}) \iff o(s, m) = \text{yes} \tag{10}$$

$$s \in \kappa_S(\mathbf{a}^\perp) \iff o(s, m) = \text{no} \tag{11}$$

Similar equations apply for the state $\alpha(m)$ with yes and no switched.

Arguably, the ultimate purpose of observation is not only to measure a single property, but to infer the state of an entity by performing tests and formulating outcomes. Such an observer is able to observe the state of the entity S and will be called “ S -knowledgeable.” In the operational approach to quantum logic (Aerts, 1999), the so-called “state determination axiom” tells us that a state is determined by the properties that are actual in that state. If an observer is to be able to infer an arbitrary state of S , then for each separate actual property \mathbf{a} of the entity S that he is asked to observe, he should be able to acquire a state m that is \mathbf{a} -perfect. We formulate this in our last definition.

Definition 10. An observer M will be said to be S -knowledgeable iff for an arbitrary state $s \in \Sigma_S$ of S and for every property $\mathbf{a} \in \mathcal{L}_S$ that is actual for s , there exists a state $m \in \Sigma_M$ that is \mathbf{a} -perfect.

Note that this definition is lenient in the sense that it does not require there should exist a *single* state of the observer that is perfect for all properties. However, this advantage automatically disappears when the observer examines a property of himself, because perfect observation of a property entails that he gives the right answer, regardless of the state of the system under investigation!

2.2. Undecidability in Observation

Can an observer M find out in which state a given system S is? By definition he can iff for every property \mathbf{a} that is actual in state s , there exists a state $m \in \Sigma_M$ that is \mathbf{a} -perfect. Obviously, the set of actual properties always includes the classical properties of S , because a property is classical iff the property or its inverse is actual. Let us concentrate on a specific classical property \mathbf{a} of S . An observer could well be \mathbf{a} -classical perfect, or he might not be. In this way we define a new candidate property for M : the property of “ \mathbf{a} -classical perfectness,” which we will denote by $\mathbf{p}_\mathbf{a}$. The test that corresponds to this property—letting the observer interact with S and verify whether he perfectly observes the property \mathbf{a} —can be reliably performed only by an absolute observer. But even if such an absolute observer exists, how is the observer to know for himself whether he is perfect or not with respect to the observation of a classical property? He cannot rely on the outcome given by another “god-like” observer, because such an outcome is yet another example of a classical property that he needs to observe, giving rise to

the same problem. If the observer is to find out, he will have to rely on his own power of observation. For example, he should at least be sure about the outcome he observed, i.e., about the actuality of the classical indicator property \mathbf{i} . We now show there are fundamental restrictions to the observation of classical properties that pertain to himself.

Theorem 3. *Let \mathbf{a} be a classical property of M . If M is \mathbf{a} -classical perfect, then $\mathbf{p}_\mathbf{a}$ is classical too.*

Proof: Because \mathbf{a} is a property of M , and by the definition of perfect observation, M can only be \mathbf{a} -classical perfect if M can tell for any state in Σ_M whether \mathbf{a} is actual or not. Hence all states of M have to be \mathbf{a} -classical perfect. Furthermore, because \mathbf{a} is a classical property, \mathbf{a} -classical perfect observation entails that the outcome has to be predetermined. A predetermined outcome as a result of an observation of a classical property, is either always right or always wrong, so that either $\mathbf{p}_\mathbf{a}$ or $\mathbf{p}_\mathbf{a}^\perp$ is actual for all $m \in \Sigma_M$. \square

Theorem 4. *Let \mathbf{a} be a classical property of M . If M is \mathbf{a} -classical perfect, then M cannot observe $\mathbf{p}_\mathbf{a}$ -classical perfectly.*

Proof: We proceed ad absurdum and assume that there exists a nonempty set of states denoted $\Sigma_M^{\mathbf{p}_\mathbf{a}}$, such that $m \in \Sigma_M^{\mathbf{p}_\mathbf{a}}$ observes $\mathbf{p}_\mathbf{a}$ -classical perfect, i.e., m observes $\mathbf{p}_\mathbf{a}$ -perfectly, and $\alpha(m)$ observes $\mathbf{p}_\mathbf{a}^\perp$ -perfectly. To answer the question whether an arbitrary state m is $\mathbf{p}_\mathbf{a}$ -classical perfect, M investigates the state m (either by introspection or by examining an identical system in the same state). Because the property tested is $\mathbf{p}_\mathbf{a}$ —which is classical by Theorem 3—the definition of $\Sigma_M^{\mathbf{p}_\mathbf{a}}$ is obtained by rewriting (10) and (11) for M, m and $\mathbf{p}_\mathbf{a}$:

$$\Sigma_M^{\mathbf{p}_\mathbf{a}} = \{m \in \Sigma_M : m \in \kappa_M(\mathbf{p}_\mathbf{a}) \iff o(m, m) = \text{yes}\} \tag{12}$$

$$\Sigma_M^{\mathbf{p}_\mathbf{a}} = \{m \in \Sigma_M : m \in \kappa_M(\mathbf{p}_\mathbf{a}^\perp) \iff o(m, m) = \text{no}\} \tag{13}$$

But classical perfectness with respect to $\mathbf{p}_\mathbf{a}$, by inverse completeness, entails that there exists at least one $\alpha(m)$ in Σ_M with $\alpha(m) \in \kappa_M(\mathbf{p}_\mathbf{a}^\perp)$. Hence, if M is in the state $\alpha(m)$, application of (13) to $\alpha(m)$ gives:

$$\alpha(m) \in \kappa_M(\mathbf{p}_\mathbf{a}^\perp) \iff o(\alpha(m), \alpha(m)) = \text{no} \tag{14}$$

But if Eq. (14) holds for $\alpha(m)$, then by (13), we have $\alpha(m) \in \Sigma_M^{\mathbf{p}_\mathbf{a}}$. This means $\mathbf{p}_\mathbf{a}$ is actual for $\alpha(m)$ and $\alpha(m) \in \kappa_M(\mathbf{p}_\mathbf{a})$. By (12), the outcome should have been yes. \square

The construction of the proof relies on the necessary existence of $\alpha(m)$ and is therefore recognized as a diagonal argument. For such an $\alpha(m)$ we have to assume

the outcome is “no,” expressing he is not \mathbf{p}_a -perfect. But then this was a perfect observation and it should have been “yes.” The structure that we find when an observer attempts to answer the question of his own nonperfectness, is similar to the well-known Liar paradox, or the Gödel sentence “ $x: x$ is not provable,” whose very proof would seem to imply the truth of the proposition, which states that it is not provable, and so on. Regarded as a logical proposition, the terminology to indicate this logically circular decision problem was called “undecidable” by Gödel (1931), hence the title of this paper. As a consequence of Theorem 4, we now prove that no observer can observe his own state perfectly.

Theorem 5. *No observer M can be M -knowledgeable.*

Proof: An observer is M -knowledgeable if he can perfectly observe *all* actual properties of the state m he is in. Suppose \mathbf{a} is a classical property of M . There is at least one such \mathbf{a} , because we postulated the outcome indicator is a classical property. If M is not \mathbf{a} -classical perfect, then he cannot know his own state. Hence we assume M is \mathbf{a} -classical perfect. By Theorem 3, \mathbf{p}_a is classical too, and he needs to be able to observe that property classical perfectly. We have shown that he cannot, indicating he cannot observe all his actual properties. □

3. CONCLUDING REMARKS

The real problem is, of course, that all observation is self-observation. The detector doesn’t measure an exterior system directly, but rather through an act of observation in the changes in the state of its own system. Of course, the argument doesn’t deny that real observers *can* make Λ -perfect observations with a high probability of success for a variety of properties. The result says only that M cannot observe whether his observation was Λ -perfect, or not. Also, nothing in our argument denies the possibility of a second observer observing this first observer to be perfect (or not!) in his observation. This would preempt the self-referential loop in the proof. But this second observer faces the same problem, leading to an infinite regression that cannot solve the original problem. This reminds one of the way a stronger formal system can be used to decide whether a given formal system is complete and consistent, but even this stronger formal system cannot decide its own completeness and consistency. One could say that the property of perfectness is potential only. This stance is viable but begs the question how we should observe if we cannot do it perfectly. Another possible generalization is to allow for a countable set of outcomes for a test. It is, however, a quite characteristic feature of undecidability arguments, that the essential result does not depend on the cardinality of the outcome set, as long as it is countable. That no observer can observe its own state perfectly, seems in close accordance with Breuer’s results (Breuer, 1995), in which Breuer has shown

by a very elegant argument that there exist different states of an observer that he himself cannot distinguish. To the best of our knowledge, the first presentation showing the relevance of these issues in relation to the quantum measurement problem, is the trail-blazing 1977 paper by Dalla Chiara (1977). This is not the place to attempt an overview of the extensive literature on the subject, and we refer to Busch *et al.* (1991); Svozil (1996) and the references therein. In a sense, these results explain why the quantum measurement problem has not been solved. If the theory describes every fundamental interaction, then it also has to describe the process of observation. This allows for self-reference because the theory talks about the way the results are tested, which are produced by that same theory. Self-reference in turn, allows for the undecidability. An undecidable proposition cannot be made decidable by additional knowledge because it is not an epistemological issue. Thus an ontological uncertainty about at least some of the measurement outcomes becomes unavoidable. In agreement with Breuer (1995), we argue this is not necessarily due to the nonclassicality of the properties or theory, but because we regard the theory as fundamental, i.e., describing *all* processes. Perhaps one should not describe the process of observation, or describe it in a fashion entirely different than other processes. This pragmatic stance is taken in present day quantum physics, and as a result, we have two different, incompatible evolution laws and no clear rule to tell us what precisely constitutes an observation and what is a normal interaction, and why they should be treated differently. It seems we are left with two logical alternatives⁵ for any theory that includes a description of the observation of the quantities it predicts: either we have a dichotomic split between the process of observation and other interactions, or we include both under a single heading and face the undecidability. In an upcoming article we will argue that the second possibility can serve as an alternative formulation of the basic structure of quantum probability.

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⁵ It was kindly pointed out by a referee that a third option would be that a system does not have actual properties.

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